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# Acoustic 'black holes' for flexural waves as effective vibration dampers

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**Abstract**

Elastic plates of variable thickness gradually decreasing to zero (elastic wedges) can support a variety of unusual effects for flexural waves propagating towards sharp edges of such structures and reflecting back. Especially interesting phenomena may take place in the case of plate edges having cross-sections described by a power law relationship between the local thickness $h$ and the distance from the edge $x$: $h(x) = \varepsilon x^m$, where $m$ is a positive rational number and $\varepsilon$ is a constant. In particular, for $m \geqslant 2$—in free wedges, and for $m \geqslant 5/3$—in immersed wedges, the incident flexural waves become trapped near the edge and do not reflect back, i.e., the above structures represent acoustic 'black holes' for flexural waves. However, because of the ever-present edge truncations in real manufactured wedges, the corresponding reflection coefficients are always far from zero. The present paper shows that the deposition of absorbing thin layers on the plate surfaces can dramatically reduce the reflection coefficients. Thus, the combined effect of the specific wedge geometry and of thin absorbing layers can result in very efficient damping systems for flexural vibrations.


## 1. Introduction

It is well known that when flexural vibrations (flexural elastic waves) propagate towards the edges of elastic plates of variable thickness gradually decreasing to zero (elastic wedges), they slow down and their amplitudes grow as they approach the edges. After reflection from the edge, with the module of reflection coefficient normally being equal to unity, the whole process repeats in the opposite direction [1,2]. If waves propagate obliquely from sharp edges towards wedge foundations, the corresponding gradual increase in local flexural wave velocity may cause total

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internal refraction of the incident flexural waves. The internally refracted flexural waves then reflect from the edge and experience internal refraction again, thus causing waveguide propagation along wedge edges—the so-called wedge acoustic waves (see, e.g., Refs. [1–5]).

Especially interesting phenomena may occur in the special case of wedges having cross-sections described by a power law relationship between the local thickness $h$ and the distance from the edge $x$: $h(x) = \varepsilon x^m$ (see Fig. 1), where $m$ is a positive rational number and $\varepsilon$ is a constant [4–6]. In particular, for $m \geqslant 2$—in free wedges, and for $m \geqslant 5/3$—in immersed wedges, the flexural waves incident at an arbitrary angle upon a sharp edge can become trapped near the very edge and therefore never reflect back [5,6]. Thus, such structures materialize acoustic 'black holes', if one uses the analogy with corresponding astrophysical objects. In the case of localized flexural waves (also known as wedge acoustic waves) propagating along wedge edges of power-law profile the phenomenon of acoustic 'black holes' implies that wedge acoustic wave velocities in such structures become equal to zero [4,5]. This reflects the fact that the incident wave energy becomes trapped near the edge and wedge acoustic waves simply do not propagate.

The 'black hole' effects are not only known for flexural waves in elastic wedges. As has been predicted theoretically by several authors, the principal possibility of the effects of zero reflection can exist also for wave phenomena of a different physical nature. In particular, this may occur for underwater sound propagation in a layer with sound velocity profile linearly decreasing to zero with increasing depth [7]. Similarly, the reflection may be absent for internal waves in a horizontally inhomogeneous stratified fluid [8,9] and for particle scattering in quantum mechanics [10]. For seismic interface waves propagating in soft marine sediments with power-law shear speed exponent equal to unity it is the wave velocity that may be equal to zero [11,12], exactly as in the case of the above-mentioned wedge acoustic waves in elastic wedges with power-law profile [4,5].

One must note, however, that, whereas the conditions providing zero wave reflection can rarely be found in real ocean environment or for real atomic potentials, wedges of arbitrary power-law

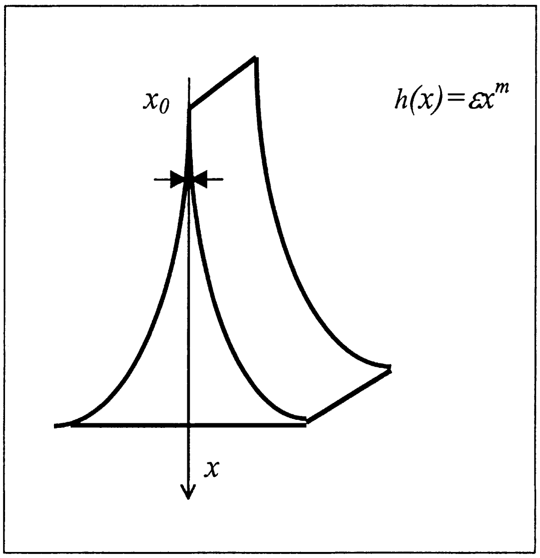


Fig. 1. Elastic wedge of power-law profile.

profile are relatively easy to manufacture. Thus, elastic solid wedges give the unique opportunity to materialize the above-mentioned zero-reflection effects normally associated with 'black holes' and to use them for practical purposes.

The unusual effect of power-law profile on flexural wave propagation in elastic wedges has attracted some initial attention in respect of their possible applications as vibration absorbers. Mironov [6] was the first to point out that a flexural wave does not reflect from the edge of a square-shaped wedge in vacuum ($m = 2$), so that even a negligibly small material attenuation may cause all the wave energy to be absorbed near the edge. Unfortunately, because of the deviations of manufactured wedges from the ideal power-law shapes, largely due to ever-present truncations of the wedge edges, real reflection coefficients in such homogeneous wedges are always far from zero [6]. Therefore, in practice such wedges can not be used as vibration absorbers.

In the present paper, it is demonstrated that the situation can be radically improved by modifying the wedge surfaces. In particular, the presence of thin absorbing layers on the surfaces of wedges with power-law profile can drastically reduce the reflection coefficients. Thus, the combination of the effects of the specific wedge geometry and of thin absorbing layers can result in very efficient damping systems for flexural vibrations.

## 2. Geometrical acoustics approach

To explain the basic principle of the phenomenon consider propagation of plane flexural waves (or bending waves) towards the edge of a free slender wedge of arbitrary shape. Flexural wave propagation in such wedges can be described using the geometrical acoustics approach considering a slender wedge as a plate with a variable local thickness $h(x)$ (see Refs. [1–5] for more detail). The starting point for developing this approach is a two-dimensional equation for bending motions of a thin plate in vacuum, with the plate local thickness $h$ being variable along the $x$ direction:

$$\frac{\partial^2}{\partial x^2}\left[D(x)\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\sigma)\frac{\partial^2}{\partial x\,\partial y}\left[D(x)\frac{\partial^2 w}{\partial x\,\partial y}\right]+\frac{\partial^2}{\partial y^2}\left[D(x)\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right)\right]-\omega^2\rho h(x)w=0. \quad (1)$$

Here $w$ is the normal displacement of the median plane of the plate, $D(x) = Eh^3(x)/12(1-\sigma^2) = (\rho c_p^2/12)h^3(x)$ is the local bending stiffness, $E$ and $\sigma$ are Young's modulus and the Poisson ratio of the plate material, $\rho$ is its mass density, $c_p = 2c_t(1 - c_t^2/c_l^2)^{1/2}$ is the velocity of quasi-longitudinal waves in thin plates (or plate velocity), $c_l$ and $c_t$ are longitudinal and shear wave velocities in the plate material, and $\omega = 2\pi f$ is circular frequency.

The solution of Eq. (1) in the usual geometrical-acoustics (or optics) representation is sought:

$$w = A(x)\exp[ik_pS(x,y)], \quad (2)$$

where $A(x)$ and $S(x,y) = S'(x) + (\beta/k_p)y$ are the slowly varying amplitude and *eiconal* of the quasi-plane wave, respectively, $\beta$ is the conserved projection of the wave vector of this wave onto the $y$ axis (for normal incidence of the wave $\beta = 0$), and $k_p = \omega/c_p$ is the wave number of quasi-longitudinal waves in thin plates.

Substituting Eq. (2) into Eq. (1), one has to equalize both real and imaginary parts of the resulting expression to zero. Equating the real part to zero and retaining only the leading terms, one can derive the so-called 'eiconal equation' for flexural waves [1,2]:

$$|\nabla S(x,y)|^4 = k^4(x)/k_p^4 = n^4(x), \tag{3}$$

where $k(x) = 12^{1/4}k_p^{1/2}(h(x))^{-1/2}$ is the local wave number of the bending wave in the plate of variable thickness, $k_p = \omega/c_p$, $n(x)$ is the corresponding refractive index, and $\nabla = \mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y)$.

A solution of Eq. (3) corresponding to wave propagation in the positive direction along $x$-axis is

$$S'(x) = (1/k_p)\int [k^2(x) - \beta^2]^{1/2}\,\mathrm{d}x. \tag{4}$$

Equalizing the imaginary part of the resulting expression to zero, one can derive the 'transport equation' which describes evolution of the amplitude $A(x)$ while the wave propagates towards the edge. This equation is rather cumbersome and it is, therefore, not reproduced here (for more detail see Refs. [1,2]). It can be shown that the function $A(x)$ obtained from the solution of the transport equation satisfies the energy conservation law for bending waves propagating through various cross-sections of the plate of variable thickness [1,2], which gives the alternative method of calculating the amplitudes $A(x)$ for waves propagating in wedges of different profiles.

A thorough estimate of the terms which have been discarded in deriving the above-mentioned eiconal and transport equations has been done in the paper [1] for the case of a linear wedge characterized by the wedge angle $\Theta$: $h(x) = \Theta x$. It has been shown that the above-mentioned geometrical-acoustics solutions for $S'(x)$ and $A(x)$ are valid under the condition $k_p x/\Theta \gg 1$. In other words, this solution breaks down in the immediate vicinity of the wedge edge (at small values of $k_p x$) and/or at large wedge angles $\Theta$. Note that the condition $k_p x/\Theta \gg 1$, which has been derived in Ref. [1] as a result of direct analysis of the fourth order differential equation for plate flexural vibrations (Eq. (1)), can also be formally obtained from the well known applicability condition of classical geometrical acoustics (or optics)

$$\left|\frac{1}{k^2}\frac{\mathrm{d}k}{\mathrm{d}x}\right| \ll 1 \tag{5}$$

if it is applied to $k(x) = 12^{1/4}k_p^{1/2}(h(x))^{-1/2}$, where $h(x) = \Theta x$. Condition (5) follows from the analysis of the second order Helmholtz equation in layered media, and it has been also used in Ref. [6], albeit without proof, for analysis of flexural wave propagation in wedges of power law profile. Apparently, condition (5) is the most general applicability condition of geometrical acoustics approximation that is applicable to waves of different physical nature, regardless of the type of wave equation. For that reason, in further consideration of flexural wave propagation in elastic wedges of arbitrary profile condition (5) only will be used.

## 3. Wedges of power-law profile

### 3.1. General consideration

Consider flexural wave propagation in the normal direction (see formula (2) with $\beta = 0$) towards the edge of a wedge having a power-law profile: $h(x) = \varepsilon x^m$. It follows from Eqs. (2) and (4) that the integrated wave phase $\Phi = k_p S(x)$ resulting from the wave propagation from an arbitrary initial point $x$ taken on the wedge medium plane to the wedge tip ($x = 0$) can be written in the form

$$\Phi = \int_0^x k(x)\,\mathrm{d}x. \tag{6}$$

Here $k(x)$ is the local wave number of a flexural wave propagating in a wedge contacting with vacuum. For a wedge of power-law profile this local wave number can be written in the form

$$k(x) = 12^{1/4} k_p^{1/2} (\varepsilon x^m)^{-1/2}. \tag{7}$$

From Eqs. (6) and (7) one can easily see that the integral in Eq. (6) diverges for $m \geqslant 2$. This means that the phase $\Phi$ becomes infinite under these circumstances, which implies that the wave never reaches the edge. Therefore, it never reflects back either, i.e., the wave becomes trapped, thus indicating that the above mentioned ideal wedges represent acoustic 'black holes' for incident flexural waves.

Real fabricated wedges, however, always have different imperfections, mainly truncated edges. And this affects adversely their performance as potential vibration absorbers. If for ideal wedges of power-law shape (with $m \geqslant 2$) it follows from Eqs. (6) and (7) that even an infinitely small material attenuation, that can be described by adding an imaginary part to $k(x)$ in Eq. (7), would be sufficient for the total wave energy to be absorbed, this is not so for truncated wedges. Indeed, for truncated wedges the lower integration limit in Eq. (6) must be changed from 0 to a certain value $x_0$ describing the length of the truncation. This results in the amplitude of the total reflection coefficient $R_0$ being expressed in the form [6]

$$R_0 = \exp\left(-2 \int_{x_0}^x \mathrm{Im}\, k(x)\,\mathrm{d}x\right), \tag{8}$$

which takes into account flexural wave propagation from the point $x$ to the truncation point $x_0$, the 100% 'local' reflection of the flexural wave energy from the free truncated edge and propagation of the reflected wave back from $x_0$ to $x$. According to Eq. (8), for typical values of attenuation in the wedge materials, even very small truncations $x_0$ result in the total reflection coefficients $R_0$ becoming as large as tens per cent.

To improve the situation for real wedges with edge truncations, now consider covering the wedge surfaces by thin absorbing layers (films) of thickness $\delta$ , e.g., by polymeric films (see Fig. 2). Note in this connection that the idea of applying absorbing layers for damping flexural vibrations of uniform plates is not new and has been successfully used since the 1950s (see, e.g., Refs. [13–15]). The new aspect of this idea, which is being explored in the present paper, is the use of such absorbing layers in combination with a specific geometry of a plate of variable thickness (a wedge) to achieve maximum damping.

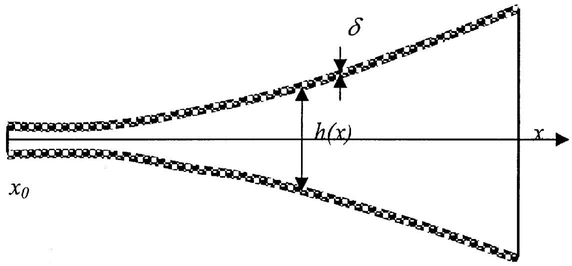


Fig. 2. Truncated wedge covered by thin absorbing layers.

In what follows only one possible film-induced attenuation mechanism is considered—the one associated with in-plane deformations of the film (layer) under impact of flexural waves. Such deformations occur on the wedge surfaces as a result of the well-known relationship between flexural displacements $u_z$ and longitudinal displacements $u_x$ in a plate: $u_x = -z(\partial^2 u_z/\partial x^2)$. Not specifying the physical mechanism of the damping in the film material, it is assumed for simplicity that, as happens in many practical cases, it is linearly dependent on frequency, with non-dimensional constant $v$ being the energy loss factor, or simply the loss factor.

To analyze the effect of thin absorbing films on flexural wave propagation in a wedge in the framework of geometrical acoustics approximation one should first consider the effect of such films on flexural wave propagation in plates of constant thickness. The latter problem can be approached in different ways. For example, it can be solved using the non-classical boundary conditions taking into account the so-called 'surface effects' [16,17]. Alternatively, the energy perturbation method developed by Auld [18] can be used. However, the simplest way is to use the already known solutions for plates covered by absorbing coatings of arbitrary thickness obtained by different authors with regard to the description of damped vibrations in such sandwich plates [13–15].

In particular, for a plate of constant thickness $h$ covered by a visco-elastic layer of thickness $\delta$ on one of the surfaces the following expression for the additional loss factor $\xi$ can be obtained [13–15]:

$$\xi = \frac{v}{[1 + (\alpha_2\beta_2(\alpha_2^2 + 12\alpha_{21}^2))^{-1}]}. \tag{9}$$

Here $v$ is the loss factor of the material of the visco-elastic layer, $\alpha_2 = \delta/h$, $\beta_2 = E_2/E_1$, and $\alpha_{21} = (1 + \alpha_2)/2$, where $E_1$ and $E_2$ are, respectively, Young's moduli of the plate and of the visco-elastic layer. Assume now that the plate is covered by visco-elastic layers on both surfaces and consider the limiting case of $\alpha_2 = \delta/h \ll 1$. Then, assuming that $\alpha_2\beta_2 \ll 1$, one can arrive from Eq. (9) to the following simplified expression:

$$\xi = 6\alpha_2\beta_2 v = 6(\delta/h)(E_2/E_1)v. \tag{10}$$

Using Eq. (10), one can write down the expression that takes into account the effects of both thin absorbing layers and of the wedge material and geometry on the imaginary part of a flexural wave number, $\mathrm{Im}\,k(x)$, for a wedge of non-linear shape characterized by the local thickness $h(x)$:

$$\mathrm{Im}\,k(x) = \left[\frac{12^{1/4}k_p^{1/2}}{h^{1/2}(x)}\right]\left[\frac{\eta}{4} + \frac{3}{2}\frac{\delta}{h(x)}\frac{E_2}{E_1}v\right], \tag{11}$$

where $\eta$ is the loss factor of the wedge material. The additional flexural wave attenuation caused by the absorbing film and described by the second term in Eq. (11) is proportional to the ratio of the film thickness $\delta$ to the plate thickness $h$, and to the ratio of Young's moduli, $E_2/E_1$, of the film and plate, respectively.

### 3.2. Quadratic wedges

Consider a wedge of quadratic shape, i.e., with a local thickness $h(x) = \varepsilon x^2$. Substituting Eq. (11) into Eq. (8) and performing the integration, one can obtain the following analytical solution for the resulting total reflection coefficient $R_0$:

$$R_0 = \exp(-2\mu_1^{(2)} - 2\mu_2^{(2)}). \tag{12}$$

Here

$$\mu_1^{(2)} = \frac{12^{1/4}k_p^{1/2}\eta}{4\varepsilon^{1/2}}\ln\left(\frac{x}{x_0}\right) \tag{13}$$

and

$$\mu_2^{(2)} = \frac{3 \times 12^{1/4}k_p^{1/2}v\delta}{4\varepsilon^{3/2}}\frac{E_2}{E_1}\frac{1}{x_0^2}\left(1 - \frac{x_0^2}{x^2}\right). \tag{14}$$

In the absence of the absorbing film ($\delta = 0$ or $v = 0$, and hence $\mu_2^{(2)} = 0$), Eqs. (12)–(14) reduce to the results obtained in Ref. [6] (where the typographical misprint has been observed). If the absorbing film is present ($\delta \neq 0$ and $v \neq 0$), this brings the additional reduction of the reflection coefficient that depends on the film loss factor $v$ and on the other geometrical and physical parameters of the wedge and of the film.

Now consider the applicability of geometrical acoustics approximation for wedges of power-law profile. It follows from Eqs. (5) and (7) that the corresponding applicability condition is

$$\frac{\varepsilon^{1/2}mx^{(m-2)/2}}{2 \times 12^{1/4}k_p^{1/2}} \ll 1. \tag{15}$$

For quadratic wedges ($m = 2$) it follows from Eq. (15) that geometrical acoustics approximation is valid for all $x$ provided that the following inequality is satisfied:

$$\frac{12^{1/4}k_p^{1/2}}{\varepsilon^{1/2}} \gg 1. \tag{16}$$

For a majority of practical situations this condition can be easily satisfied even at relatively low frequencies. For example, if $\varepsilon = 0.05\,\mathrm{m}^{-1}$ and $c_t = 2000\,\mathrm{m/s}$, then condition (16) implies that frequency $f = \omega/2\pi$ should be much larger (e.g., by four times larger) than say 20 Hz. As will be seen in the following section, this low-frequency limitation arising from the applicability condition of geometrical acoustics is not very essential since, because of the linear frequency dependence of material attenuation, the wedge damping structure under consideration is most efficient at relatively high frequencies.

For illustration purposes let the following values be chosen of the film parameters: $\nu = 0.2$ (i.e., consider the film as being highly absorbing), $E_2/E_1 = 2/3$ and $\delta = 5\,\mu\text{m}$. Let the parameters of the quadratically shaped wedge be: $\varepsilon = 0.05\,\text{m}^{-1}$, $\eta = 0.01$, $x_0 = 1.5\,\text{cm}$, $x = 50\,\text{cm}$ and $c_p = 3000\,\text{m/s}$. Then, e.g., for the frequency $f = 10\,\text{kHz}$ it follows from Eqs. (12) to (14) that in the presence of the absorbing film $R_0 = 0.017$ (i.e., 1.7%), whereas in the absence of the absorbing film $R_0 = 0.513$ (i.e., 51.3%). Thus, in the presence of the absorbing film the value of the reflection coefficient is much smaller than for a wedge with the same value of truncation, but without a film. Even relatively high absorption in the wedge material ($\eta = 0.01$) only slightly reduces the reflection coefficient in an uncovered wedge from unity corresponding to the hypothetical case of a wedge made of non-dissipative material.

It is interesting to compare the reflection coefficient of flexural waves for a wedge covered by a thin absorbing film with the one for the same waves reflecting from the blunt edge of a homogeneous plate made of the same material and having the thickness ($h = 1.3\,\text{cm}$) equal to the initial local thickness $h_{ini}$ of the wedge considered in the above example ($h_{ini} = 1.3\,\text{cm}$ at $x = 50\,\text{cm}$). Calculations show that in the case of such a homogeneous plate $R_0 = 0.832$, i.e., the reflection coefficient for the plate is still very large and only slightly differs from the reflection coefficient for the plate without an absorbing film ($R_0 = 0.834$). Thus, the effect of thin absorbing film causes in this case only a slight reduction of the reflection coefficient from its value defined by energy losses in a plate. Obviously, it is both the unusual geometrical properties of a quadratically shaped wedge in respect of wave propagation and the effect of thin absorbing layers that result in a very efficient way of suppressing flexural vibrations.

Note that almost all absorption of the incident wave energy takes place in the vicinity of the sharp edge of a wedge. Fig. 3 shows the reflection coefficient $R_0$ for the wedge considered as function of the total wedge length $x$ for the two values of the wedge edge truncation $x_0$. It is seen that $R_0$ changes noticeably only in the close proximity of the wedge edge. In the example discussed above (with $\nu = 0.2$), almost 99% of the incident elastic energy is absorbed within the length of 3 cm near the truncated edge.

The influence of length of the wedge edge truncation $x_0$ on the reflection coefficient $R_0$ is shown on Fig. 4 for an uncovered wedge with the parameters described above and for the same wedge covered by absorbing films with the same values of the film thickness ($\delta = 5\,\mu\text{m}$) and film material loss factor ($\nu = 0.2$), but with the different values of relative film stiffness ($E_2/E_1 = 2/30$ and $2/3$). One can see that the values of $x_0$ still retaining the reflection coefficients $R_0$ that are close to zero depend strongly on the film relative stiffness. The larger relative stiffness the larger values of truncation $x_0$ can be allowed. One should keep in mind, however, that Eqs. (12)–(14) have been derived under the assumption of the film thickness being much less than the local thickness of the wedge. Therefore, strictly speaking, calculations on Fig. 4 are not valid for $x_0$ close to zero, where the film thickness becomes comparable and even larger than the wedge local thickness.

Fig. 5 illustrates the frequency dependence of the resulting reflection coefficient $R_0$ for the two values of edge truncation $x_0 = 1.5$ and $2.5\,\text{cm}$ in wedges covered by absorbing films and in uncovered wedges. As one can see, in all cases the reflection coefficients tend to zero with the increase of frequency. Although for wedges covered by thin absorbing films such a tendency is much more rapid, the considered wedge structures in general seem to be efficient as dampers only

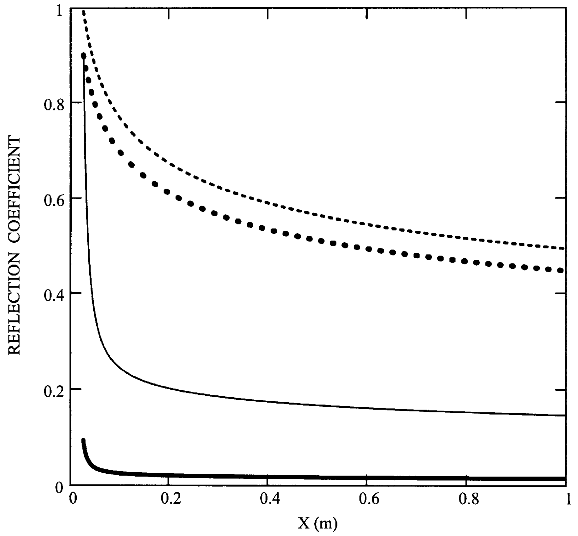


Fig. 3. Reflection coefficient $R_0$ as a function of the total wedge length $x$ (in $m$) for the values of wedge edge truncation $x_0 = 1.5\,\text{cm}$ (thicker curves) and $x_0 = 2.5\,\text{cm}$ (thinner curves): solid and dotted curves correspond to wedges with and without absorbing layers, respectively; the film material loss factor $\nu$ is 0.2, and the film thickness $\delta$ is $5\,\mu\text{m}$.

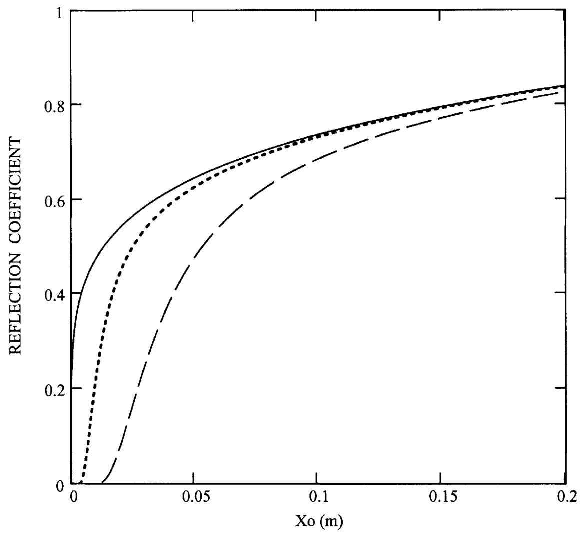


Fig. 4. Effect of wedge truncation $x_0$ (in $m$) on the reflection coefficient $R_0$: solid curve corresponds to an uncovered wedge, dotted and dashed curves correspond to wedges covered by thin absorbing films with the values of relative stiffness $E_2/E_1 = 2/30$ and $E_2/E_1 = 2/3$, respectively; the film material loss factor $\nu$ is 0.2, and the film thickness $\delta$ is $5\,\mu\text{m}$.

at relatively high frequencies (higher than 2–3 kHz). Therefore, the geometrical acoustics approximation that has been used in the previous section for their theoretical description is adequate (see Eq. (16)).

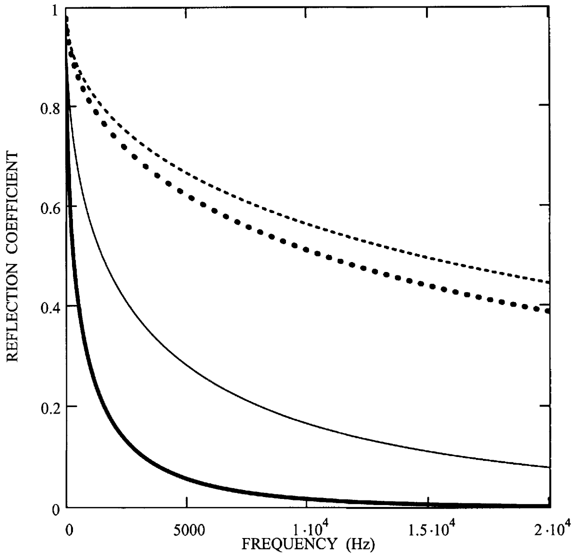


Fig. 5. Frequency dependence of the reflection coefficient $R_0$ for two values of the edge truncation: $x_0 = 1.5$ and 2.5 cm (thicker and thinner curves respectively); solid and dotted curves correspond to wedges with absorbing films and to uncovered wedges; the film material loss factor $\nu$ is 0.2, and the film thickness $\delta$ is 5 μm.

For very low frequencies, when flexural wavelengths are comparable with the total wedge length $x$, the theoretical model under consideration becomes inaccurate also because of another reason. Strictly speaking, in this case one should take into account non-propagating exponentially decaying flexural waves that may be excited at the wedge foundation by the vibrating structure to be damped. However, since at low frequencies the reflection coefficients from the wedge edge are far from zero even in the assumption that only propagating waves are reaching the wedge edge, taking into account non-propagating waves may prove practically unimportant.

### *3.3. Wedges of power-law profiles with m=3 and 4*

In this section the effects of power-law profiles are considered with higher values of $m$ on the reflection coefficients of flexural waves from the wedge edges. Performing the integration in Eq. (8) for the function $h(x) = \varepsilon x^m$ with $m = 3$, one can obtain:

$$R_0 = \exp(-2\mu_1^{(3)} - 2\mu_2^{(3)}), \tag{17}$$

where

$$\mu_1^{(3)} = \frac{12^{1/4}\eta k_p^{1/2}}{2\varepsilon^{1/2}} \frac{1}{x_0^{1/2}} \left[1 - \left(\frac{x_0}{x}\right)^{1/2}\right] \tag{18}$$

and

$$\mu_2^{(3)} = \frac{3 \times 12^{1/4}\delta \nu k_p^{1/2}}{7\varepsilon^{3/2}} \frac{E_2}{E_1} \frac{1}{x_0^{7/2}} \left[1 - \left(\frac{x_0}{x}\right)^{7/2}\right]. \tag{19}$$

Similarly, carrying out the integration in Eq. (8) for the function $h(x) = \varepsilon x^m$ with $m = 4$, one can get:

$$R_0 = \exp(-2\mu_1^{(4)} - 2\mu_2^{(4)}), \tag{20}$$

where

$$\mu_1^{(4)} = \frac{12^{1/4}\eta k_p^{1/2}}{4\varepsilon^{1/2}} \frac{1}{x_0}\left[1 - \left(\frac{x_0}{x}\right)\right] \tag{21}$$

and

$$\mu_2^{(4)} = \frac{3 \times 12^{1/4}\delta \nu k_p^{1/2}}{10\varepsilon^{3/2}} \frac{E_2}{E_1} \frac{1}{x_0^5}\left[1 - \left(\frac{x_0}{x}\right)^5\right]. \tag{22}$$

Numerical calculations for these profiles have been carried out using the same wedge material and absorbing film parameters that had been used in the calculations for a wedge with a quadratic profile. Namely, for the film: $\nu = 0.2$, $E_2/E_1 = 2/3$ and $\delta = 5\,\mu\text{m}$; for the wedge: $\varepsilon = 0.05\,\text{m}^{-1}$, $\eta = 0.01$, $x_0 = 1.5\,\text{cm}$, $x = 50\,\text{cm}$ and $c_p = 3000\,\text{m/s}$, $f = 10\,\text{kHz}$. The results of calculations of the reflection coefficients $R_0$ as functions of the edge truncation $x_0$ for wedges covered by absorbing films and for uncovered wedges are shown on Fig. 6 for $m = 3$ and 4. For comparison, the earlier obtained results for $m = 2$ are shown in Fig. 6 as well.

It can be seen from Fig. 6 that for $m = 3$ and 4 the trend followed by the reflection coefficient $R_0$ is similar to that for $m = 2$. However, comparing the curves for $m = 3$ and 4 with that for $m = 2$, one can see that there is a remarkable improvement in the reduction of the reflection coefficient for larger values of $m$.

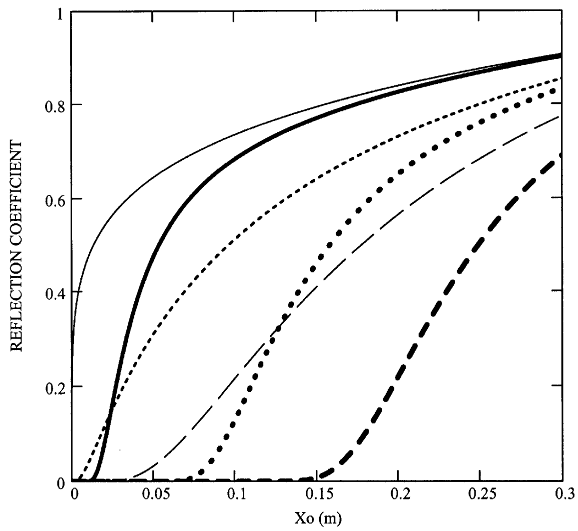


Fig. 6. Effect of edge truncation $x_0$ (in $m$) on the reflection coefficient $R_0$ for wedges of power-law profile with $m = 2$, 3 and 4 (solid, dotted and dashed curves respectively): thicker curves correspond to wedges covered by thin absorbing films (with $\nu = 0.2$, $\delta = 5\,\mu\text{m}$, $E_2/E_1 = 2/3$) and thinner curves correspond to uncovered wedges.

Considering the conditions of geometrical acoustics approximation for power-law-profiled wedges with $m = 3$ and 4, one can derive from Eq. (15) the following applicability conditions for $m = 3$ and 4 respectively:

$$\frac{3\varepsilon^{1/2}x^{1/2}}{2 \times 12^{1/4}k_p^{1/2}} \ll 1, \tag{23}$$

$$\frac{2\varepsilon^{1/2}x}{12^{1/4}k_p^{1/2}} \ll 1. \tag{24}$$

As it follows from Eqs. (23) and (24), for the above wedges the geometrical acoustics approximation becomes invalid for large distances $x$ from the wedge edges.

Despite the very positive results obtained for wedges with $m = 3$ and 4, it has to be noted that in these cases the wedge edge profiles become even sharper, making it extremely difficult to manufacture such wedges. Also, as in the case of quadratic wedges, one of the assumptions which no longer holds in the case of the higher powers of $m$ is that for very small truncations $x_0$ the absorbing film remains thin in comparison with the wedge local thickness, ideally at least four times as thin. Therefore, these assumptions have to be verified each time to ensure that wedges of power-law profile with higher powers of $m$ are viable.

## 4. Wedges of sinusoidal profile

In this section a new type of function for a wedge profile is explored that also results in acoustic 'black hole' effects, similar to those obtained for wedges with power-law profiles. A search of different possibilities has yielded a function $h(x) = \varepsilon \sin^m(x)$ for $m \geqslant 2$. This choice can be explained by the fact that for small values of $x$, the function $\varepsilon \sin^m(x)$ can be approximated as a power-law profile $\varepsilon x^m$, which has been analyzed in the previous sections. For clarity, consider functions $h(x) = \varepsilon \sin^m x$ for $m = 2$ and 4.

Performing the integration in Eq. (8) for $h(x) = \varepsilon \sin^m(x)$ with $m = 2$, one can obtain

$$R_0 = \exp(-2\mu_1^{(s2)} - 2\mu_2^{(s2)}), \tag{25}$$

where

$$\mu_1^{(s2)} = \frac{12^{1/4}\eta k_p^{1/2}}{4\varepsilon^{1/2}}\left[\ln\frac{(\csc x - \cot x)}{(\csc x_0 - \cot x_0)}\right] \tag{26}$$

and

$$\mu_2^{(s2)} = \frac{3 \times 12^{1/4}\delta\nu k_p^{1/2}}{4\varepsilon^{3/2}}\frac{E_2}{E_1}\left(\ln\frac{(\csc x - \cot x)}{(\csc x_0 - \cot x_0)} + \frac{\cos x_0}{\sin^2 x_0} - \frac{\cos x}{\sin^2 x}\right). \tag{27}$$

Similarly, integrating in Eq. (8) for $h(x) = \varepsilon \sin^m(x)$ with $m = 4$, one can get

$$R_0 = \exp(-2\mu_1^{(s4)} - 2\mu_2^{(s4)}), \tag{28}$$

where

$$\mu_1^{(s4)} = \frac{12^{1/4}\eta k_p^{1/2}}{4\varepsilon^{1/2}}\left(\frac{\cos x_0}{\sin x_0} - \frac{\cos x}{\sin x}\right) \tag{29}$$

and

$$\mu_2^{(s4)} = \frac{12^{1/4}\delta v k_p^{1/2}}{10\varepsilon^{3/2}}\frac{E_2}{E_1}\left(3\left(\frac{\cos x_0}{\sin^5 x_0} - \frac{\cos x}{\sin^5 x}\right) + 4\left(\frac{\cos x_0}{\sin^3 x_0} - \frac{\cos x}{\sin^3 x}\right) + 8\left(\frac{\cos x_0}{\sin x_0} - \frac{\cos x}{\sin x}\right)\right). \tag{30}$$

Now the conditions are discussed under which the geometrical acoustics approximation is valid for the above-mentioned sinusoidal functions $h(x) = \varepsilon \sin^m(x)$. Using Eq. (5), one can find that geometrical acoustics approximation is valid for wedges of sinusoidal profiles if

$$\frac{\varepsilon^{1/2}}{2 \times 12^{1/4} k_p^{1/2}} m \sin^{(m-2)/2}(x)\cos(x) \ll 1. \tag{31}$$

For $m = 2$ this condition reduces to the inequality

$$\frac{12^{1/4} k_p^{1/2}}{\varepsilon^{1/2}\cos(x)} \gg 1, \tag{32}$$

which differs from the corresponding applicability condition for a quadratic wedge (see Eq. (16)) only by the presence of the function $\cos(x)$ in the denominator. Since $|\cos(x)| \leqslant 1$ this only improves the applicability of geometrical acoustics approximation for sinusoidal wedges which, as well as in the case of quadratic wedges, is applicable at all distances $x$.

For $m = 4$ it follows from Eq. (31) that the applicability condition in this case is

$$\frac{12^{1/4} k_p^{1/2}}{\varepsilon^{1/2}\sin(2x)} \gg 1. \tag{33}$$

Comparing this condition with the one for a power-law-profiled wedge with $m = 4$ (see Eq. (24)), one can see that in the corresponding case of sinusoidal profile the applicability is significantly improved and no longer restricted to the area adjacent to the edge.

The results of calculations of the reflection coefficients of flexural waves for wedges of sinusoidal profiles as functions of the truncation length $x_0$ are shown on Fig. 7. For comparison, on the same figure the reflection coefficients for wedges of the corresponding power-law profiles are shown as well. As expected, the results for wedges of sinusoidal profiles with $m = 2$ and 4 are very similar, and sometimes almost identical, to those exhibited by wedges of power-law profiles with the same powers $m = 2$ and 4. This perfect match of the results is due to the previously mentioned reason that at small values of $x$, which are the most influential, both types of profiles are approximately identical. As in the case of power-law profiles, the allowable values of wedge truncation $x_0$ for sinusoidal profiles are much larger in the case of higher values of $m$.

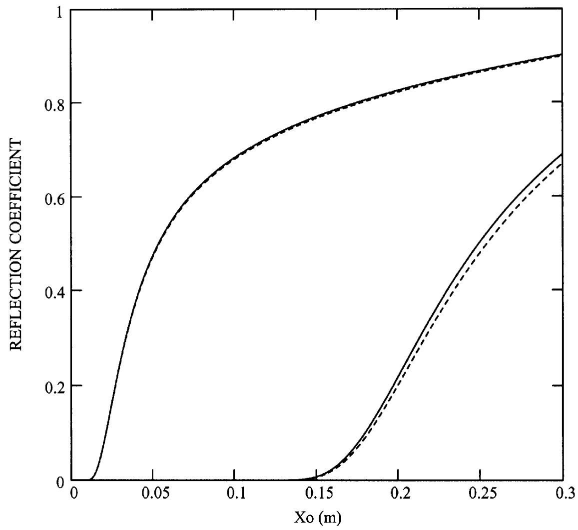


Fig. 7. Effect of edge truncation $x_0$ (in $m$) on the reflection coefficient $R_0$ for wedges of sinusoidal and power-law profiles with $m = 2$ and 4 covered by thin absorbing films with the parameters $\nu = 0.2$, $\delta = 5\,\mu\text{m}$, $E_2/E_1 = 2/3$: solid curves correspond to wedges of power-law profiles, and dotted curves—to wedges of sinusoidal profiles; two almost coinciding curves on the left correspond to $m = 2$, and two curves on the right—to $m = 4$.

## 5. Conclusions

Some theoretical results have been reported on the new type of vibration absorbers utilizing the effect of acoustic 'black holes' for flexural waves propagating in elastic wedges of power-law and associated sinusoidal profiles. It has been demonstrated that the presence of thin absorbing films on the surfaces of such wedges can significantly reduce the reflection of flexural waves from their truncated edges due to the enhanced energy absorption by such films.

For more rigorous evaluation of flexural wave reflection in wedges of non-linear shape covered by thin absorbing films, more elaborate physical models of wave energy absorption by thin films should be considered (e.g., the ones accounting for viscous friction, relaxation-type attenuation mechanisms, etc.). Although the corresponding estimates of the reflection coefficients may differ significantly for different film attenuation mechanisms, it is expected that in all cases it will be possible to achieve very significant reductions in values of the reflection coefficients $R_0$. The reason for this is that it is the combination of the extraordinary wave propagation properties of power-law-shaped and sinusoidal wedges and the effect of thin absorbing layers that reduces dramatically the wave reflection coefficient (as compared to a free wedge and to a homogeneous plate covered by an absorbing film). This can open a very efficient way of suppressing flexural vibrations.

The advantage of using the proposed wedge absorbers of flexural vibrations over traditional types of vibration absorbers lays in the fact that wedge absorbers are compact and light-weight since they do not require additional masses to provide resonant absorption. Apart from this, they can be integrated into structures to be damped on a design stage, as their inseparable parts.

Despite the very encouraging theoretical estimates described above, further theoretical and experimental investigations are needed to validate the principle and to explore the most efficient ways of creating the above-mentioned 'wedge-like absorbers' of flexural vibrations.

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